

Characterizations of the upper bound of Bakry-Emery curvature*

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Abstract

In this paper, we will present some characterizations for the upper bound of the Bakry-Emery curvature on a Riemannian manifold by using functional inequalities on path space. Moreover, some characterizations for general lower and upper bounds of Ricci curvature are also given, which extends the recent results derived by Naber [19] and Wang-Wu[25].

Keywords: Bakry-Emery Curvature; functional inequality, diffusion process, path space.

1 Introduction

Let (M, g) be an n -dimension complete Riemannian manifold and Z be a C^1 -vector field on M . Consider the Bakry-Emery curvature $\text{Ric}_Z := \text{Ric} + \nabla Z$ for the Witten Laplace $L := \Delta_M - Z$, where Δ_M is the Laplace operator on M . The bounded property of Ric_Z play a very important role on the field of analysis. Specially, there are a number of equivalent characterizations for the lower bound of Ric_Z by functional inequalities of the (Neumann) semigroup generated by L in [22]. However, for the associated upper bound, there are not almost any results. In this paper, some equivalent characterizations for the upper bound of Ric_Z will be presented, and we will also give some characterizations for general lower and upper bounds of Ric_Z . These results extends the recent results derived by Naber [19] and Wang-Wu[25]. The motivation of this work comes from the

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following observations: The upper and lower bounds of the Bakry-Emery curvature Ric_Z at any point $x \in M$ are actually determined by the distribution properties of all paths near this point.

Before moving on, let us recall some notation on path space. For any $T > 0$ and $x \in M$, the based path space and the path space over M are defined respectively by

$$W^T(M) := C([0, T]; M)$$

and

$$W_x^T(M) := \{\gamma \in W^T(M) : \gamma_0 = x\}.$$

Let d be the distance function on M . Then $W^T(M)$ is a Polish space under the uniform distance

$$\rho(\gamma, \sigma) := \sup_{t \in [0, T]} d(\gamma_t, \sigma_t), \quad \gamma, \sigma \in W^T(M).$$

In particular, let $\rho_x(\gamma) := \rho(\gamma, x)$, $\gamma \in W^T(M)$ be the distance function starting from some fixed $x \in M$ on $W^T(M)$.

Denote by $O_x(M)$ be the orthonormal frame bundle at $x \in M$, then $O(M) := \sup_{x \in M} O_x(M)$ is the orthonormal frame bundle over M . Let U_t^x be the horizontal diffusion process on $O(M)$ generated by L ; that is, U_t^x solves the following stochastic differential equation on $O(M)$,

eq1.1

$$(1.1) \quad dU_t^x = \sqrt{2} \sum_{i=1}^d H_i(U_t^x) \circ dW_t^i + H_Z(U_t^x) dt, \quad U_0 \in O_x(M),$$

where $W_t = (W_t^1, \dots, W_t^d)$ is the d -dimensional Brownian motion, $\{H_i\}_{i=1}^n : TM \rightarrow TO(M)$ is a standard orthonormal basis of horizontal vector fields on $O(M)$ and H_Z is the horizontal vector fields of Z . Let $\pi : O(M) \rightarrow M$ be the canonical projection. Then $X_t^x := \pi(U_t^x)$, $t \geq 0$ is the L -diffusion process starting from x on M .

For any $T > 0$, define the Cameron-Martin space

$$\mathbb{H} = \left\{ h \in C([0, T]; \mathbb{R}^d) : h(0) = 0, \|h\|_{\mathbb{H}}^2 := \int_0^T |h'_s|^2 ds < \infty \right\},$$

which is a separable Hilbert space under $\langle h, g \rangle_H := \int_0^T \langle h'_s, g'_s \rangle ds$, $h, g \in \mathbb{H}$. Here $\mathcal{F}C_T$ denotes the class of cylindric smooth functions on the path space $W^T(M)$ defined as

eq1.2

$$(1.2) \quad \mathcal{F}C_T := \left\{ F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_N)) : N \geq 1, \gamma \in W^T(M), \right. \\ \left. 0 < t_1 < t_2 < \dots < t_N \leq T, f \in C_0^{Lip}(M^N) \right\}.$$

For each $F \in \mathcal{F}C_T$ with the form $F(\gamma) := f(\gamma(t_1), \dots, \gamma(t_N))$ and any $h \in \mathbb{H}$, the Malliavin derivative $D_h F$ is given by

$$\text{eq1.3} \quad (1.3) \quad D_h F(X_{[0,T]}^x) = \sum_{i=1}^N \langle \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x), U_{t_i}^x h(t_i) \rangle_{T_{X_{t_i}^x} M},$$

where ∇_i is the (distributional) gradient operator for the i -th component on M^N . By the Riesz representation theorem, there exists a gradient function $DF \in \mathbb{H}$ such that

$$\langle DF(X_{[0,T]}^x), h \rangle_{\mathbb{H}} = D_h F(X_{[0,T]}^x), \quad h \in \mathbb{H}.$$

In particular, if F has the above form, we have

$$\text{eq1.4} \quad (1.4) \quad \dot{D}_s F(X_{[0,T]}^x)(s) := \frac{d[DF(X_{[0,T]}^x)(s)]}{ds} = \sum_{t_i > s} (U_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x).$$

On behalf of stating our main result, we also introduce some new notation. For any $x \in M$ and $R > 0$, denote by $B_R(x)$ the unit closed ball with x as the center and R as the radius, and the local cylindric functions is defined by

$$\text{eq1.5} \quad (1.5) \quad \mathcal{F}C_{loc}^{x,R,T} = \left\{ f(\gamma(t_1), \dots, \gamma(t_N)) l(\rho_x(\gamma)) : \right. \\ \left. 0 < t_1 < \dots < t_N < T, f \in C_0^{Lip}(M^N), l \in C_o^\infty(\mathbb{R}), \text{supp}(l) \subset B_R(x) \right\}.$$

Let K_1, K_2 be any two continuous functions on M with $K_1 \geq K_2$, we introduce the following random measure on $[0, T]$:

$$\text{MU} \quad (1.6) \quad \mu_{x,T}^{K_1, K_2}(ds) := e^{\frac{\int_0^s K_1(X_u^x) - K_2(X_u^x) du}{2}} \frac{K_1(X_s^x) - K_2(X_s^x)}{2} ds$$

and the measurable function on $W^T(M)$:

$$\text{eq1.7} \quad (1.7) \quad A_{s,t}^{K_1, K_2}(X_{[0,T]}^x) := e^{-\frac{\int_s^t K_1(X_u^x) + K_2(X_u^x) du}{2}}.$$

For simplicity, let $A_t^{K_1, K_2}(X_{[0,T]}^x) := A_{0,t}^{K_1, K_2}(X_{[0,T]}^x)$. According to the Riesz representation theorem, for every Lipschitz function F on $W(M)$, there exists an unique gradient $D^{K_1, K_2} F$ such that

$$D_h F(X_{[0,T]}^x) = \langle D^{K_1, K_2} F(X_{[0,T]}^x), A^{K_1, K_2}(X_{[0,T]}^x) h \rangle_{\mathbb{H}}, \quad \mathbb{P} - \text{a.s.}$$

In particular, if $F \in \mathcal{F}C_T$ with the form $F(\gamma) = f(\gamma(t_1), \dots, \gamma(t_N))$, then

$$\text{MG} \quad (1.8) \quad \dot{D}_s^{K_1, K_2} F(X_{[0,T]}^x) = \sum_{t_i > s} A_{t_i}^{K_1, K_2}(X_{[0,T]}^x) (U_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x), \quad s \in [0, T].$$

and $D^{K_1, K_2} F \in \mathbb{H}$. In fact,

$$\begin{aligned}
\langle D^{K_1, K_2} F, h \rangle_{\mathbb{H}} &= \int_0^T \langle \dot{D}_s^{K_1, K_2} F, h'_s \rangle ds \\
&= \sum_{i=1}^N A_{t_i}^{K_1, K_2} \langle (U_{t_i}^x)^{-1} \nabla_i f, h_{t_i} \rangle = \sum_{i=1}^N \left\langle (U_{t_i}^x)^{-1} \nabla_i f, A_{t_i}^{K_1, K_2} h_{t_i} \right\rangle \\
&= \sum_{i=1}^N \int_0^{t_i} \left\langle (U_{t_i}^x)^{-1} \nabla_i f, \left(A_s^{K_1, K_2} h_s \right)' \right\rangle ds \\
&= \int_0^T \left\langle \sum_{i=1}^N 1_{\{s \leq t_i\}} (U_{t_i}^x)^{-1} \nabla_i f, \left(A_s^{K_1, K_2} \right)' \right\rangle ds \\
&= \int_0^T \left\langle \dot{D}_s F, \left(A_s^{K_1, K_2} h_s \right)' \right\rangle ds = \langle DF, A^{K_1, K_2} h \rangle_{\mathbb{H}}.
\end{aligned}$$

Specially, when $K_2 = -K_1$, we have $D^{K_1, K_2} F = DF$.

For any $t \in [0, T]$, define respectively the gradient

eq1.9

$(1.9) \quad \dot{D}_{t,s}^{K_1, K_2} F(X_{[0,T]}^x) = \sum_{t_i > s} A_{t_i}^{K_1, K_2} (X_{[0,T]}^x) (U_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x), s \geq t.$

and the energy form

$$\begin{aligned}
\mathcal{E}_{t,T}^{K_1, K_2}(F, F) &= \mathbb{E} \left\{ \left(1 + \mu_{x,T}^{K_1, K_2}([t, T]) \right) \left(\left| \dot{D}_{t,t}^{K_1, K_2} F(X_{[0,T]}^x) \right|^2 \right. \right. \\
&\quad \left. \left. + \int_t^T \left| \dot{D}_{t,s}^{K_1, K_2} F(X_{[0,T]}^x) \right|^2 \mu_{x,T}^{K_1, K_2}(ds) \right) \right\}
\end{aligned}$$

for each Lipschitz function F on $W(M)$. For any $x \in M, R > 0$, let

$$C_R^x := \inf \{ \text{Ric}_Z(X, X) : X \in T_y M, |X| = 1, y \in B_R(x) \}.$$

Our main results are as below:

T1.1 **Theorem 1.1.** *Let K be a continuous function on M . The following statements are equivalent each other:*

(1) *For any $x \in M$,*

$$\text{Ric}_Z(x) \leq K(x)g_x.$$

(2) *For any $x \in M$ and for any constants $R > 0, C \leq C_R^x$ (or there exist constants $R > 0$ and $C \leq C_R^x$), and for each $T > 0$,*

$$\left| \nabla_x \mathbb{E} F(X_{[0,T]}^x) \right| \leq \mathbb{E} \left\{ \left| \dot{D}_0^{K, C} F \right| + \int_0^T \left| \dot{D}_s^{K, C} F(X_{[0,T]}^x) \right| \mu_{x,T}^{K, C}(ds) \right\}, \quad F \in \mathcal{F}C_{loc}^{x, R, T}.$$

- (3) For any $x \in M$ and for any constants $R > 0, C \leq C_R^x$ (or there exist constants $R > 0$ and $C \leq C_R^x$), and for each $T > 0$,

$$|\nabla_x \mathbb{E} F(X_{[0,T]}^x)|^2 \leq \mathbb{E} \left\{ (1 + \mu_{x,T}^{K,C}([0, T])) \times \left(|\dot{D}_0^{K,C} F|^2 + \int_0^T |\dot{D}_s^{K,C} F(X_{[0,T]}^x)|^2 \mu_{x,T}^{K,C}(ds) \right) \right\}, \quad F \in \mathcal{F}C_{loc}^{x,R,T}.$$

- (4) For any $x \in M$ and for any constants $R > 0, C \leq C_R^x$ (or there exist constants $R > 0$ and $C \leq C_R^x$), and for each $T > 0$ and for any $t_0, t_1 \in [0, T]$ with $t_1 > t_0$, the following log-Sobolev inequality holds:

$$\mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1})] - \mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0})] \leq 4 \int_{t_0}^{t_1} \mathcal{E}_{s,T}^{K,C}(F, F) ds, \quad F \in \mathcal{F}C_{loc}^{x,R,T}.$$

- (5) For any $x \in M$ and for any constants $R > 0, C \leq C_R^x$ (or there exist constants $R > 0$ and $C \leq C_R^x$), and for each $T > 0$ and for any $t_0, t_1 \in [0, T]$ with $t_1 > t_0$, the following Poincaré inequality holds:

$$\mathbb{E} \left[\left\{ \mathbb{E}(F(X_{[0,T]}^x) | \mathcal{F}_{t_1}) \right\}^2 \right] - \left\{ \mathbb{E}[F(X_{[0,T]}^x)] \right\}^2 \leq 2 \int_0^{t_1} \mathcal{E}_{s,T}^{K,C}(F, F) ds, \quad F \in \mathcal{F}C_{loc}^{x,R,T}.$$

Remark 1.1. We remark that the L -diffusion process may be explosive in Theorem 1.1.

Next, the following Theorem 1.2 will characterize general lower and upper bounds of the Bakry-Emery curvature.

T1.2 **Theorem 1.2.** Suppose K_1 and K_2 are two continuous functions on M with $K_1 \geq K_2$ such that

DE (1.10) $\mathbb{E} e^{(2+\varepsilon) \int_0^T |K_1|(X_s^x) + |K_2|(X_s^x) ds} < \infty$ for some $\varepsilon, T > 0$.

For any $p, q \in [1, 2]$, the following statements are equivalent each other:

- (1) For any $x \in M$,

$$K_2(x)g_x \leq \text{Ric}_Z(x) \leq K_1(x)g_x.$$

- (2) For any $T > 0$ and $x \in M$, $f \in C_0^\infty(M)$ with $|\nabla f|(x) = 1$,

$$\begin{aligned} |\nabla P_T f|^p(x) &\leq \mathbb{E} \left[e^{-p \int_0^T K_2(X_u^x) du} |\nabla f|^p(X_T^x) \right], \\ \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^q &\leq \mathbb{E} \left[\left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right)^{q-1} \right. \\ &\quad \times \left. \left(\left| \nabla f(x) - \frac{1}{2} A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^q + \frac{\mu_{x,T}^{K_1, K_2}([0, T])}{2^q} \left(A_T^{K_1, K_2} \right)^q |\nabla f(X_T^x)|^q \right) \right]. \end{aligned}$$

(3) For any $F \in \mathcal{F}C_T$, $x \in M$ and $T > 0$,

$$|\nabla_x \mathbb{E} F(X_{[0,T]}^x)|^q \leq \mathbb{E} \left\{ \left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right)^{q-1} \times \left(|\dot{D}_0^{K_1, K_2} F|^q + \int_0^T \left| \dot{D}_s^{K_1, K_2} F(X_{[0,T]}^x) \right|^q \mu_{x,T}^{K_1, K_2}(ds) \right) \right\}.$$

(4) For any $t_0, t_1 \in [0, T]$ with $t_1 > t_0$, and any $x \in M$, the following log-Sobolev inequality holds:

$$\begin{aligned} & \mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1})] \\ & - \mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0})] \leq 4 \int_{t_0}^{t_1} \mathcal{E}_{s,T}^{K_1, K_2}(F, F) ds, \quad F \in \mathcal{F}C_T. \end{aligned}$$

(5) For any $t \in [0, T]$ and $x \in M$, the following Poincaré inequality holds:

$$\mathbb{E} \left[\left\{ \mathbb{E}(F(X_{[0,T]}^x) | \mathcal{F}_t) \right\}^2 \right] - \left\{ \mathbb{E}[F(X_{[0,T]}^x)] \right\}^2 \leq 2 \int_0^t \mathcal{E}_{s,T}^{K_1, K_2}(F, F) ds, \quad F \in \mathcal{F}C_T.$$

Remark 1.2. (1) When $K_2(x) = -K_1(x)$, it is proved in [25, Theorem 1.1] that $\|\text{Ric}_Z\|_\infty \leq K_1(x)$ is equivalent to each of (3)-(5) and a slightly different formulation for $\dot{D}_s^{K_1, K_2} F$.

(2) According to Theorem 1.2, M is an Einstein manifold with $\text{Ric} = K$ if and only if all/some of items (2)-(5) hold for $K_1 = K_2 = K$ and $Z = 0$.

(3) When we obtained Theorem 1.2, we also heard from Cheng-Thalmaier[8] that they had given some characterizations for the lower and upper bounds of Ricci curvature by using two different functional inequalities. Their results are similar to ours, but the techniques are also somewhat different. Two papers were finished independently.

The rest of this paper is organized as follows: In Section 2, we will present the proof of Theorem 1.2. As an application, some equivalent conditions for variable lower bounds of Ricci curvature will be given. Finally, complete proof of Theorem 1.1 will be outlined in Section 3.

2 Proof of Theorem 1.2

In this section, we will mainly prove Theorem 1.2. To do that, we introduce some basic notation partly coming from [24]. Let $f \in C_0^\infty(M)$ with $|\nabla f(x)| = 1$ and $\text{Hess}_f(x) = 0$. According to [24, Theorem 3.2.3], if $x \in M$ then for any $p > 0$ we have

$$\begin{aligned} \text{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{t \downarrow 0} \frac{P_t |\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt} \\ \text{RIC} \quad (2.1) \quad &= \lim_{t \downarrow 0} \frac{1}{t} \left(\frac{P_t f^2(x) - (P_t f)^2(x)}{2t} - |\nabla P_t f(x)|^2 \right). \end{aligned}$$

For any $s \geq 0$, consider the following resolvent equation

$$\boxed{\text{eq2.3}} \quad (2.2) \quad \frac{dQ_{s,t}^x}{dt} = -Q_{s,t}^x \text{Ric}_Z(U_t^x) \quad t \geq s, \quad Q_{s,s}^x = \text{Id}.$$

Then $(Q_{s,t}^x)_{t \geq s}$ is an adapted right-continuous process on $\mathbb{R}^d \otimes \mathbb{R}^d$. For the sake of convenience, let $Q_t^x := Q_{0,t}^x$. By [17, Proposition 2.1], for any $F \in \mathcal{F}C_T^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \cdot, \gamma_{t_N})$, $f \in C_0^{Lip}(M)$ and $0 \leq t_1 < \dots \leq t_N$,

$$\boxed{\text{eq2.2}} \quad (2.3) \quad (U_0^x)^{-1} \nabla_x \mathbb{E}[F(X_{[0,T]}^x)] = \sum_{i=1}^N \mathbb{E} \left[Q_{t_i}^x (U_{t_i}^x)^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x) \right],$$

where ∇_x denotes the gradient in $x \in M$ and ∇_i is the gradient with respect to the i -th component. In particular, taking $F(\gamma) = f(\gamma_t)$, we obtain

$$\boxed{\text{GR2}} \quad (2.4) \quad \nabla P_t f(x) = U_0^x \mathbb{E} [Q_t^x (U_t^x)^{-1} \nabla f(X_t^x)], \quad x \in M, f \in C_0^\infty(M), t \geq 0.$$

Finally, for the above $F \in \mathcal{F}C_T$, the damped gradient of F is denoted by

$$\boxed{\text{TTD}} \quad (2.5) \quad \tilde{D}_t F(X_{[0,T]}^x) = \sum_{i: t_i > t} Q_{t,t_i}^x U_{t_i}^{-1} \nabla_i f(X_{t_1}^x, \dots, X_{t_N}^x), \quad t \in [0, T].$$

Then the martingale theorem in [4](see also [23, 24]) implies that

$$\boxed{\text{MF}} \quad (2.6) \quad \mathbb{E}(F(X_{[0,T]}^x) | \mathcal{F}_t) = \mathbb{E}[F(X_{[0,T]}^x)] + \sqrt{2} \int_0^t \langle \mathbb{E}(\tilde{D}_s F(X_{[0,T]}^x) | \mathcal{F}_s), dW_s \rangle, \quad t \in [0, T].$$

Proof of Theorem 1.2. For simplicity, below we will write F and f for $F(X_{[0,T]}^x)$ and $f(X_{t_1}^x, \dots, X_{t_N}^x)$ respectively. If Ric_Z has symmetric lower and upper bounds, A. Naber [19] and Wang-Wu[25] proved the results for the constant bound and for pointwise symmetric bound respectively. But when the lower and upper bounds of Ric_Z is not symmetric, it is difficult to establish some functional inequalities by using the uniform norm of Ric_Z such that these inequalities can characterise the associated the lower and upper bounds. To overcome the difficulty, we may make a symmetrization of Bakry-Emery curvature such that it is symmetric, i.e. we may consider the curvature

$$\text{Ric}_Z^{K_1, K_2}(U_t^x) := \text{Ric}_Z(U_t^x) - \frac{K_1(X_t^x) + K_2(X_t^x)}{2} \text{Id},$$

then by (1), we get

$$\boxed{\text{eq2.7}} \quad (2.7) \quad \left\| \text{Ric}_Z^{K_1, K_2}(U_t^x) \right\| \leq \frac{K_1(X_t^x) - K_2(X_t^x)}{2}.$$

Let $\tilde{Q}_{s,t}^x$ be the solution of the following resolvent equation

$$\boxed{\text{eq2.8}} \quad (2.8) \quad \frac{d\tilde{Q}_{s,t}^x}{dt} = -\tilde{Q}_{s,t}^x \text{Ric}_Z^{K_1, K_2}(U_t^x), \quad t \geq s, \quad \tilde{Q}_{s,s}^x = \text{Id}.$$

and denote $\tilde{Q}_t^x := \tilde{Q}_{0,t}^x$ for convenience. Combining this with (2.8),

$$\boxed{\text{eq2.9}} \quad (2.9) \quad \tilde{Q}_{s,t}^x = e^{\frac{\int_s^t K_1(X_s^x) + K_2(X_s^x) ds}{2}} Q_{s,t}^x.$$

Following the line of [25], in the following, we will present a full proof of Theorem 1.2.

(a) (1) \Rightarrow (3) for all $q \geq 1$. According to (2.3), we have

$$U_0^{-1} \nabla_x \mathbb{E}[F] = \mathbb{E} \left[\sum_{i=1}^N Q_{t_i}^x (U_{t_i}^x)^{-1} \nabla_i f \right].$$

Then by (2.9) and (1.7),

$$\begin{aligned} \boxed{\text{eq2.10}} \quad (2.10) \quad & U_0^{-1} \nabla_x \mathbb{E}[F] = \mathbb{E} \left[\sum_{i=1}^N \tilde{Q}_{t_i}^x A_{t_i}^{K_1, K_2} (U_{t_i}^x)^{-1} \nabla_i f \right] \\ &= \mathbb{E} \left[\sum_{i=1}^N \left(I + \int_0^{t_i} \tilde{Q}_s^x \left(\text{Ric}_Z(U_s^x) - \frac{K_1(X_t^x) + K_2(X_t^x)}{2} \text{Id} \right) ds \right) A_{t_i}^{K_1, K_2} (U_{t_i}^x)^{-1} \nabla_i f \right] \\ &= \mathbb{E} \left[\sum_{i=1}^N A_{t_i}^{K_1, K_2} (U_{t_i}^x)^{-1} \nabla_i f \right. \\ &\quad \left. + \sum_{i=1}^N \left(\int_0^{t_i} \tilde{Q}_s^x \left(\text{Ric}_Z(U_s^x) - \frac{K_1(X_t^x) + K_2(X_t^x)}{2} \text{Id} \right) ds \right) A_{t_i}^{K_1, K_2} (U_{t_i}^x)^{-1} \nabla_i f \right] \\ &= \mathbb{E} \left[\dot{D}_0^{K_1, K_2} F + \int_0^T \left\{ \tilde{Q}_s^x \left(\text{Ric}_Z(U_s^x) - \frac{K_1(X_t^x) + K_2(X_t^x)}{2} \text{Id} \right) \dot{D}_s^{K_1, K_2} F \right\} ds \right]. \end{aligned}$$

In addition, by (2.7) and (2.8), we have

$$\boxed{\text{QQ2}} \quad (2.11) \quad \left\| \tilde{Q}_s^x \right\| \leq e^{\frac{\int_0^s K_1(X_u^x) - K_2(X_u^x) du}{2}}.$$

Combining these with (1), (1.6), and using Hölder's inequality twice, we obtain

eq2.12

$$\begin{aligned}
(2.12) \quad & |\nabla_x \mathbb{E}[F]|^q \leq \left\{ \mathbb{E} |\dot{D}_0^{K_1, K_2} F| + \mathbb{E} \int_0^T |\dot{D}_s^{K_1, K_2} F| \mu_{x, T}^{K_1, K_2}(ds) \right\}^q \\
& \leq \mathbb{E} \left\{ |\dot{D}_0^{K_1, K_2} F| + \int_0^T |\dot{D}_s^{K_1, K_2} F| \mu_{x, T}^{K_1, K_2}(ds) \right\}^q \\
& \leq \mathbb{E} \left\{ \left(|\dot{D}_0^{K_1, K_2} F|^q + \frac{(\int_0^T |\dot{D}_s^{K_1, K_2} F(X_{[0, T]}^x)| \mu_{x, T}^{K_1, K_2}(ds))^q}{\left\{ \mu_{x, T}^{K_1, K_2}([0, T]) \right\}^{q-1}} \right) (1 + \mu_{x, T}^{K_1, K_2}([0, T]))^{q-1} \right\} \\
& \leq \mathbb{E} \left\{ \left(|\dot{D}_0^{K_1, K_2} F|^q + \int_0^T |\dot{D}_s^{K_1, K_2} F(X_{[0, T]}^x)|^q \mu_{x, T}^{K_1, K_2}(ds) \right) (1 + \mu_{x, T}^{K_1, K_2}([0, T]))^{q-1} \right\}.
\end{aligned}$$

Thus, (3) holds.

(b) (3) \Rightarrow (2) for all $p = q$. Taking $F(\gamma) = f(\gamma_T)$, then we have $\mathbb{E}F(X_{[0, T]}^x) = P_T f(x)$ and

eq2.13

$$(2.13) \quad \dot{D}_s^{K_1, K_2} F = A_T^{K_1, K_2} (U_T^x)^{-1} \nabla f(X_T^x), \quad s < T.$$

Thus $|\dot{D}_s^{K_1, K_2} F| = A_T^{K_1, K_2} |\nabla f(X_T^x)|$ for $s \in [0, T]$. Finally, (3) implies the first inequality in (2) with $p = q$. Similarly, by taking $F(\gamma) = f(\gamma_0) - \frac{1}{2}f(\gamma_T)$, we have $\mathbb{E}F = f(x) - \frac{1}{2}P_T f(x)$ and

eq2.14

$$\begin{aligned}
(2.14) \quad & |\dot{D}_0^{K_1, K_2} F| = \left| \nabla f(x) - \frac{1}{2} A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|, \\
& |\dot{D}_s^{K_1, K_2} F| \leq \frac{1}{2} A_T^{K_1, K_2} |\nabla f(X_T^x)|, \quad s \in (0, T].
\end{aligned}$$

Then the second inequality in (2) is also implied by (3).

(c) (2) for some $p \geq 1$ and $q \in [1, 2] \Rightarrow$ (1). Let $x \in M$. According to the first inequality in (2) and (2.1) yield

eq2.15

$$\begin{aligned}
(2.15) \quad & 0 \leq \lim_{T \rightarrow 0} \frac{\mathbb{E} \left[e^{-p \int_0^T K_2(X_u^x) du} |\nabla f|^p(X_T^x) \right] - |\nabla P_T f|^p(x)}{pT} \\
& = \lim_{T \rightarrow 0} \mathbb{E} \left[\frac{e^{-p \int_0^T K_2(X_u^x) du} - 1}{pT} |\nabla f|^p(X_T^x) \right] + \lim_{T \rightarrow 0} \frac{P_T |\nabla f|^p(x) - |\nabla P_T f|^p(x)}{pT} \\
& = -K_2 + \text{Ric}_Z(\nabla f, \nabla f)
\end{aligned}$$

where $f \in C_0^\infty(M)$ with $\text{Hess}_f(x) = 0$ and $|\nabla f(x)| = 1$. This implies $\text{Ric}_Z \geq K_2$.

Next, we prove that the second inequality in (2) implies $\text{Ric}_Z \leq K_1$. By Hölder's inequality, the second inequality in (2) for some $q \in [1, 2]$ implies the same inequality

for $q = 2$:

$$\begin{aligned}
\text{eq2.16} \quad (2.16) \quad & \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^2 \leq \mathbb{E} \left[\left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right) \right. \\
& \times \left. \left(\left| \nabla f(x) - \frac{1}{2} A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^2 + \frac{\mu_{x,T}^{K_1, K_2}([0, T])}{4} (A_T^{K_1, K_2})^2 |\nabla f(X_T^x)|^2 \right) \right].
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\text{eq2.17} \quad (2.17) \quad & |\nabla f(x)|^2 - \langle \nabla f(x), \nabla P_T f(x) \rangle + \frac{1}{4} |\nabla P_T f(x)|^2 \\
& \leq \mathbb{E} \left[\left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right) \times \left(|\nabla f|^2(x) - \langle \nabla f(x), A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \rangle \right. \right. \\
& \quad \left. \left. + \frac{\mu_{x,T}^{K_1, K_2}([0, T]) + 1}{4} (A_T^{K_1, K_2})^2 |\nabla f|^2(X_T^x) \right) \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\text{eq2.18} \quad (2.18) \quad & 0 \leq \mathbb{E} \left[\mu_{x,T}^{K_1, K_2}([0, T]) |\nabla f|^2(x) \right. \\
& + \langle \nabla f(x), \nabla P_T f(x) - (1 + \mu_{x,T}^{K_1, K_2}([0, T])) A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \rangle \\
& + \frac{1}{4} \left[(\mu_{x,T}^{K_1, K_2}([0, T]) + 1)^2 (A_T^{K_1, K_2})^2 |\nabla f|^2(X_T^x) - |\nabla P_T f(x)|^2 \right] \\
& = \mathbb{E} \left[\mu_{x,T}^{K_1, K_2}([0, T]) |\nabla f|^2(x) + \langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1} \nabla f(X_T^x)] \rangle \right. \\
& + \left\langle \nabla f(x), \mathbb{E} \left[\left\{ 1 - (1 + \mu_{x,T}^{K_1, K_2}([0, T])) A_T^{K_1, K_2} \right\} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right] \right\rangle \\
& + \frac{1}{4} \mathbb{E} \left[\left\{ (\mu_{x,T}^{K_1, K_2}([0, T]) + 1)^2 (A_T^{K_1, K_2})^2 - 1 \right\} |\nabla f|^2(X_T^x) \right] \\
& + \frac{1}{4} [P_T |\nabla f|^2(x) - |\nabla P_T f(x)|^2].
\end{aligned}$$

By (??) and (2.4), we have

$$\begin{aligned}
\text{eq2.19} \quad (2.19) \quad & \langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1} \nabla f(X_T^x)] \rangle \\
& = - \int_0^T \langle \nabla f(x), U_0^x \text{Ric}_Z(U_r^x) (U_T^x)^{-1} \nabla f(X_T^x) \rangle dr \\
& = -T \text{Ric}_Z(\nabla f, \nabla f)(x) + o(T).
\end{aligned}$$

Combining the above these with (2.1), we obtain

$$\begin{aligned}
0 &\leq \lim_{T \rightarrow 0} \frac{\mathbb{E}[\mu_{x,T}^{K_1, K_2}([0, T])]}{T} |\nabla f(x)|^2 \\
&\quad + \lim_{T \rightarrow 0} \frac{\langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1} \nabla f(X_T^x)] \rangle}{T} \\
&\quad + \lim_{T \rightarrow 0} \frac{\langle \nabla f(x), \mathbb{E} \left[\left\{ 1 - \left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right) A_T^{K_1, K_2} \right\} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right] \rangle}{T} \\
\text{eq2.20} \quad (2.20) \quad &\quad + \frac{1}{4} \lim_{T \rightarrow 0} \frac{\mathbb{E} \left[\left\{ (\mu_{x,T}^{K_1, K_2}([0, T]) + 1)^2 (A_T^{K_1, K_2})^2 - 1 \right\} |\nabla f|^2(X_T^x) \right]}{T} \\
&\quad + \frac{1}{4} \lim_{T \rightarrow 0} \frac{[P_T |\nabla f|^2(x) - |\nabla P_T f(x)|^2]}{T} \\
&= \frac{K_1 - K_2}{2} |\nabla f(x)|^2 - \text{Ric}_Z(\nabla f, \nabla f)(x) + K_2 |\nabla f(x)|^2 \\
&\quad - \frac{K_2}{2} |\nabla f(x)|^2 + \frac{1}{2} \text{Ric}_Z(\nabla f, \nabla f)(x).
\end{aligned}$$

This implies $\text{Ric}_Z(\nabla f, \nabla f)(x) \leq K_1$.

(d) (5) \Rightarrow (1). Let $F(\gamma) = f(\gamma_T)$. Then (5) implies

$$\text{eq2.21} \quad (2.21) \quad P_T f^2(x) - (P_T f(x))^2 \leq 2 \int_0^T \mathbb{E} \left[\left(1 + \mu_{x,T}^{K_1, K_2}([s, T]) \right)^2 \left(A_{s,T}^{K_1, K_2} \right)^2 |\nabla f|^2(X_T^x) \right] ds.$$

For f in (2.1), combining this with (2.1) we obtain

$$\begin{aligned}
\text{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{T \rightarrow 0} \frac{1}{T} \left(\frac{P_T f^2(x) - (P_T f)^2(x)}{2T} - |\nabla P_T f|^2 \right) \\
&\leq \lim_{T \rightarrow 0} \frac{1}{T} \left(\frac{1}{T} \int_0^T \mathbb{E} \left[\left(1 + \mu_{x,T}^{K_1, K_2}([s, T]) \right)^2 \left(A_{s,T}^{K_1, K_2} \right)^2 |\nabla f|^2(X_T^x) \right] ds - |\nabla P_T f|^2 \right) \\
&= 2 \lim_{T \rightarrow 0} \frac{P_T |\nabla f|^2(x) - |\nabla P_T f|^2(x)}{2T} \\
&\quad + |\nabla f|^2(x) \lim_{T \rightarrow 0} \frac{\frac{1}{T} \int_0^T \mathbb{E} \left(A_{s,T}^{K_1, K_2} \right)^2 \left(1 + \mu_{x,T}^{K_1, K_2}([s, T]) \right)^2 ds - 1}{T} \\
&\leq 2 \text{Ric}_Z(\nabla f, \nabla f)(x) + |\nabla f|^2(x) \\
&\quad \times \lim_{T \rightarrow 0} \left\{ \frac{\frac{1}{T} \int_0^T \left[\mathbb{E} \left(A_{s,T}^{K_1, K_2} \right)^2 - 1 \right] ds}{T} + \frac{\frac{2}{T} \int_0^T \mathbb{E} \left[\left(A_{s,T}^{K_1, K_2} \right)^2 \mu_{x,T}^{K_1, K_2}([s, T]) \right] ds}{T} \right\} \\
&= 2 \text{Ric}_Z(\nabla f, \nabla f)(x) + (-K_1 - K_2 + K_1 - K_2) |\nabla f(x)|^2 \\
&= 2 \text{Ric}_Z(\nabla f, \nabla f)(x) - 2K_2 |\nabla f|^2(x).
\end{aligned}$$

This implies $\text{Ric}_Z(\nabla f, \nabla f)(x) \geq K_2 |\nabla f(x)|^2$.

Next, we will prove the upper bound estimates. We take $F(\gamma) = f(\gamma_\varepsilon) - \frac{1}{2}f(\gamma_T)$ for $\varepsilon \in (0, T)$. By (1.8),

$$\begin{aligned} |\dot{D}_{t,s}^{K_1, K_2} F| &= \left| A_{t,\varepsilon}^{K_1, K_2} \nabla f(X_\varepsilon) - \frac{1}{2} A_{t,T}^{K_1, K_2} U_\varepsilon^x (U_T^x)^{-1} \nabla f(X_T^x) \right| 1_{[0,\varepsilon)}(s) \\ &\quad + \frac{1}{2} A_{t,T}^{K_1, K_2} |\nabla f(X_T^x)| 1_{[\varepsilon, T]}(s). \end{aligned}$$

Then (5) implies

$$\begin{aligned} \text{eq2.22} \quad (2.22) \quad I_\varepsilon &:= \mathbb{E} \left[f(X_\varepsilon^x) - \frac{1}{2} \mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon) \right]^2 - \left(P_\varepsilon f(x) - \frac{1}{2} P_T f(x) \right)^2 \\ &\leq 2 \int_0^\varepsilon \mathbb{E} \left\{ \left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right) \left(\left| A_{t,\varepsilon}^{K_1, K_2} \nabla f(X_\varepsilon) - A_{t,T}^{K_1, K_2} \frac{1}{2} U_\varepsilon^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^2 \right. \right. \\ &\quad \left. \left. + |\nabla f(X_T^x)|^2 \frac{\int_\varepsilon^T (A_{\varepsilon,T}^{K_1, K_2})^2 \mu_{x,T}^{K_1, K_2}(dt)}{4} \right) \right\} dt + c\varepsilon^2 =: J_\varepsilon, \quad \varepsilon \in (0, T) \end{aligned}$$

for some constant $c > 0$. Obviously,

$$\begin{aligned} \text{eq2.23} \quad (2.23) \quad \lim_{\varepsilon \rightarrow 0} \frac{J_\varepsilon}{\varepsilon} &= \mathbb{E} \left\{ \left(1 + \mu_{x,T}^{K_1, K_2}([0, T]) \right) \left(\left| \nabla f(x) - \frac{1}{2} A_T^{K_1, K_2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^2 \right. \right. \\ &\quad \left. \left. + (A_T^{K_1, K_2})^2 \frac{\mu_{x,T}^{K_1, K_2}([0, T]) |\nabla f(X_T^x)|^2}{4} \right) \right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \text{eq2.24} \quad (2.24) \quad \frac{I_\varepsilon}{\varepsilon} &= \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} + \frac{1}{4\varepsilon} \mathbb{E} \left[\left\{ \mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon) \right\}^2 - (P_T f)^2(x) \right] \\ &\quad + \frac{\mathbb{E}[f(X_T^x) \{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon}. \end{aligned}$$

Let $f \in C_0^\infty(M)$ satisfy the Neumann boundary condition, we have

$$\text{eq2.25} \quad (2.25) \quad \lim_{\varepsilon \rightarrow 0} \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} = 2 |\nabla f|^2(x).$$

Next, (2.5) and (3.11) yield

$$\text{eq2.26} \quad (2.26) \quad \mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon) = P_T f(x) + \sqrt{2} \int_0^\varepsilon \langle \mathbb{E}(Q_{s,T}^x (U_T^x)^{-1} \nabla f(X_T^x) | \mathcal{F}_s), dW_s \rangle.$$

Then

$$\mathbb{E}[\mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon)]^2 = (P_T f)^2 + 2 \int_0^\varepsilon \mathbb{E} |Q_{0,T}^x (U_T^x)^{-1} \nabla f(X_T^x)|^2 ds.$$

This together with (2.4) leads to

$$\begin{aligned} \text{eq2.27} \quad (2.27) \quad & \lim_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon} \mathbb{E} \left[\left\{ \mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon) \right\}^2 - (P_T f)^2(x) \right] \\ &= \frac{1}{2} \left| \mathbb{E} [Q_{0,T}^x (U_T^x)^{-1} \nabla f(X_T^x)] \right|^2 = \frac{1}{2} |\nabla P_T f(x)|^2. \end{aligned}$$

Finally, by Itô's formula we have

$$\begin{aligned} P_\varepsilon f(x) - f(X_\varepsilon^x) &= P_\varepsilon f(x) - f(x) - \int_0^\varepsilon Lf(X_s^x) ds - \sqrt{2} \int_0^\varepsilon \langle \nabla f(X_s^x), U_s^x dW_s \rangle \\ &= o(\varepsilon) - \sqrt{2} \int_0^\varepsilon \langle \nabla f(X_s^x), U_s^x dW_s \rangle. \end{aligned}$$

Combining this with (2.27) and (2.28), we arrive at

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E}[f(X_T^x) \{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon} = -2 \langle \nabla f(x), \nabla P_T f(x) \rangle.$$

Substituting this and (2.27)-(2.29) into (2.26), we obtain

$$\lim_{\varepsilon \rightarrow 0} \frac{I_\varepsilon}{\varepsilon} = 2 \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^2.$$

Combining this with (2.24) and (2.25), we prove the second inequality in (2) for $q = 2$, which implies $\text{Ric}_Z \leq K_1$.

(f) (1) \Rightarrow (4). According to (3.11),

$$\text{eq2.28} \quad (2.28) \quad G_t := \mathbb{E}(F^2 | \mathcal{F}_t) = \mathbb{E}(F^2) + \sqrt{2} \int_0^t \langle \mathbb{E}(\tilde{D}_s F^2 | \mathcal{F}_s), dW_s \rangle, \quad t \in [0, T].$$

By Itô's formula,

$$\begin{aligned} \text{eq2.29} \quad (2.29) \quad & d(G_t \log G_t) = (1 + \log G_t) dG_t + \frac{|\mathbb{E}(\tilde{D}_s F^2 | \mathcal{F}_s)|^2}{G_t} dt \\ & \leq (1 + \log G_t) dG_t + 4 \mathbb{E}(|\tilde{D}_s F|^2 | \mathcal{F}_s) dt. \end{aligned}$$

Then

$$\text{eq2.30} \quad (2.30) \quad \mathbb{E}[G_{t_1} \log G_{t_1}] - \mathbb{E}[G_{t_0} \log G_{t_0}] \leq 4 \int_{t_0}^{t_1} \mathbb{E}|\tilde{D}_s F|^2 ds.$$

By (2.5) we have

$$\begin{aligned} \tilde{D}_s F &= \sum_{i=1}^N 1_{\{s < t_i\}} Q_{s,t_i}^x (U_{t_i}^x)^{-1} \nabla_i f = \sum_{i=1}^N 1_{\{s < t_i\}} A_{s,t_i}^{K_1, K_2} \tilde{Q}_{s,t_i}^x (U_{t_i}^x)^{-1} \nabla_i f \\ &= \sum_{i=1}^N 1_{\{s < t_i\}} \left(I + \int_s^{t_i} Q_{s,t}^x \left\{ \text{Ric}_Z(U_t^x) - \frac{K_1 + K_2}{2} (U_t^x) \right\} \right) A_{s,t_i}^{K_1, K_2} (U_{t_i}^x)^{-1} \nabla_i f dt \\ &= \dot{D}_{s,s}^{K_1, K_2} F + \int_s^T Q_{s,t}^x \left\{ \text{Ric}_Z(U_t^x) - \frac{K_1 + K_2}{2} (U_t^x) \right\} \dot{D}_{s,t}^{K_1, K_2} F dt. \end{aligned}$$

Combining this with (1), (2.11), and using the Schwarz inequality, we prove

$$\boxed{\text{eq2.31}} \quad (2.31) \quad |\tilde{D}_s F|^2 \leq (1 + \mu_{x,T}^{K,C}([s, T])) \left(|\dot{D}_{s,s}^{K_1, K_2} F|^2 + \int_s^T |\dot{D}_{s,t}^{K_1, K_2} F|^2 \mu_{x,T}^{K,C}(dt) \right).$$

This together with (2.30) implies the log-Sobolev inequality in (4). \square

In particular, if $K_2, K_1 \in \mathbb{R}$ are two constants with $K_1 \geq K_2$, from Theorem 1.2 we easily obtain the following Corollary 2.2.

$\boxed{\text{cor2.2}}$ **Corollary 2.1.** *Suppose $K_2, K_1 \in \mathbb{R}$ with $K_1 \geq K_2$. For any $p, q \in [1, 2]$, the following statements are equivalent each other:*

(1) *For any $x \in M$,*

$$gK_2 \leq \text{Ric}_Z(x) \leq gK_1.$$

(2) *For any $T > 0$ and $x \in M$, $f \in C_0^\infty(M)$ with $|\nabla f|(x) = 1$,*

$$\begin{aligned} |\nabla P_T f|^p(x) &\leq e^{-K_2 p T} P_T |\nabla f|^p(x), \\ \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^q &\leq e^{\frac{K_1 - K_2}{2}(q-1)T} \\ &\times \mathbb{E} \left[\left| \nabla f(x) - \frac{1}{2} e^{-\frac{K_1 + K_2}{2} T} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^q + \frac{e^{\frac{K_1 - K_2}{2} T} - 1}{2^q} e^{-\frac{K_1 + K_2}{2} q T} |\nabla f(X_T^x)|^q \right]. \end{aligned}$$

(3) *For any $F \in \mathcal{F}C_T$, $x \in M$ and $T > 0$,*

$$\begin{aligned} |\nabla_x \mathbb{E} F(X_{[0,T]}^x)|^q &\leq e^{\frac{K_1 - K_2}{2}(q-1)T} \mathbb{E} \left[|\dot{D}_0^{K_1, K_2} F(X_{[0,T]}^x)|^q \right. \\ &\quad \left. + \int_0^T |\dot{D}_s^{K_1, K_2} F(X_{[0,T]}^x)|^q \mu_{x,T}^{K,C}(ds) \right]. \end{aligned}$$

(4) *For any $t_0, t_1 \in [0, T]$ with $t_1 > t_0$, and any $x \in M$, the following log-Sobolev inequality holds:*

$$\begin{aligned} &\mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_1})] \\ &- \mathbb{E} [\mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0}) \log \mathbb{E}(F^2(X_{[0,T]}^x) | \mathcal{F}_{t_0})] \leq 4 \int_{t_0}^{t_1} \mathcal{E}_{s,T}^{K_1, K_2}(F, F) ds, \quad F \in \mathcal{F}C_T. \end{aligned}$$

(5) *For any $t \in [0, T]$ and $x \in M$, the following Poincaré inequality holds:*

$$\mathbb{E} \left[\left\{ \mathbb{E}(F(X_{[0,T]}^x) | \mathcal{F}_t) \right\}^2 \right] - \left\{ \mathbb{E}[F(X_{[0,T]}^x)] \right\}^2 \leq 2 \int_0^t \mathcal{E}_{s,T}^{K_1, K_2}(F, F) ds, \quad F \in \mathcal{F}C_T.$$

cor2.2 **Corollary 2.2.** Suppose K is a continuous function on M and

eq2.32 (2.32) $\mathbb{E}e^{-p \int_0^T K^-(X_s^x) ds} < \infty$ for some $\varepsilon, T > 0$

for some $p \geq 1$, the following two statements are equivalent:

(1) For any $x \in M$,

$$\text{Ric}_Z(x) \geq K(x).$$

(2) For any $T > 0$ and $x \in M$, $f \in C_0^\infty(M)$,

$$|\nabla P_T f|^p(x) \leq \mathbb{E} \left[e^{-p \int_0^T K(X_u^x) du} |\nabla f|^p(X_T^x) \right].$$

Proof. Obviously, (2) comes from (1) by using (2.1). In following, we only show that (1) \Rightarrow (2). Let

eq2.33 (2.33)
$$\begin{aligned} K_1(x) &= \sup \{ \text{Ric}_Z(X, X) : X \in T_x M, |X| = 1 \} \\ K_2(x) &= \inf \{ \text{Ric}_Z(X, X) : X \in T_x M, |X| = 1 \}. \end{aligned}$$

(a) If K_1 and K_2 satisfy with the integrable condition (1.10), by Theorem 1.2, then we have (3) of Theorem 1.2. In particular, for the function $F(\gamma) := f(\gamma_T)$, it implies that

eq2.34 (2.34)
$$|\nabla P_T f|(x) \leq \mathbb{E} \left[e^{-\int_0^T K_2(X_u^x) du} |\nabla f|(X_T^x) \right]$$

Thus, according to (2.32) and Cauchy-Schwartz inequality, (2) are implied by $K \leq K_2$.

(b) In general, by the similar argument in Section 3, we construct a sequence of processes $X^{x,R}$ such that $X_t^{x,R} = X_t^x$ \mathbb{P}_x -a.s. for every $t \leq \tau_R$ on M_R . Let $\text{Ric}^{(R)}, \nabla^{(R)}$ and $\|\cdot\|_\varphi$ be the associated Ricci curvature, the Levi-Civita connection, and the norm of vectors on M_φ respectively. Define K_1^R, K_2^R as (2.32) with Ric_Z replaced by $\text{Ric}_{Z^R}^{(R)}$. Then K_1^R, K_2^R satisfy with (1.10), by (a),

eq2.34 (2.35)
$$|\nabla \mathbb{E}[f(X_T^{x,R})]|^p \leq \mathbb{E} \left[e^{-p \int_0^T K_2^R(X_u^x) du} |\nabla f|^p(X_T^x) \right]$$

By (2.32), (1) and the dominated convergence theorem, we have

eq2.35 (2.36)
$$\begin{aligned} \lim_{R \rightarrow \infty} \mathbb{E} \left[e^{-p \int_0^T K_2^R(X_u^x) du} |\nabla f|^p(X_T^x) \right] &= \mathbb{E} \left[e^{-p \int_0^T K_2(X_u^x) du} |\nabla f|^p(X_T^x) \right] \\ &\leq \mathbb{E} \left[e^{-p \int_0^T K(X_u^x) du} |\nabla f|^p(X_T^x) \right] \end{aligned}$$

and

eq2.35 (2.37)
$$\begin{aligned} \lim_{R \rightarrow \infty} |\nabla \mathbb{E}[f(X_T^{x,R})]|^p &= \lim_{R \rightarrow \infty} |\mathbb{E}[Q_T^{R,*} U_T^R \nabla f(X_T^{x,R})]|^p \\ &= |\mathbb{E}[Q_T^* U_T^R \nabla f(X_T^x)]|^p = |\nabla \mathbb{E}[f(X_T^x)]|^p. \end{aligned}$$

Thus we get (2). □

3 Proof of Theorem 1.1

In this section, our main aim is to prove Theorem 1.1. In the following, we need to make some preparations. For any $R > 0$, let $\tau_R = \inf\{t \geq 0 : \rho(x, X_t^x) \geq R\}$. By [24, Lemma 3.1.1] (see also [2, Lemma 2.3]), there exists a constant $c > 0$ such that

$$\boxed{\text{L0}} \quad (3.1) \quad \mathbb{P}(\tau_R \leq T) \leq e^{-c/T}, \quad T \in (0, 1].$$

For each $T > 0$, let $\{t_i\}_{i=1}^\infty$ be the countable dense subset of the interval $[0, T]$, and define

$$\boxed{\text{c*}} \quad (3.2) \quad \rho_x^m(\gamma) := \sup_{1 \leq i \leq m} d_M(\gamma(t_i), x), \quad l_x^m(\gamma) := l(\rho_x^m(\gamma)), \quad \gamma \in W_x(M)$$

for some $l \in C_0^\infty(\mathbb{R})$. It is obvious that $l_x^m := l(\rho_x^m) \in \mathcal{F}C_b$. Note that $\rho_x^m = g\left((d_M(\gamma(t_1), x), \dots, d_M(\gamma(t_m), x))\right)$, where

$$g(s) := \max_{1 \leq i \leq m} s_i, \quad s = (s_1, \dots, s_m) \in \mathbb{R}^m,$$

since $g(s)$ is a Lipschitz continuous function on \mathbb{R}^m with Lipschitz constant 1, and $|\nabla d(o, x)| \leq 1$, then for every $m \geq 1$, we have

$$\|D^{K_1, K_2} l_x^m(\gamma)\|_{\mathbb{H}} \leq \sup_{r \in \mathbb{R}} |l'(r)| e^{\int_0^T \frac{|K_1(\gamma_s) + K_2(\gamma_s)|}{2} ds}, \quad \mu - a.s. \gamma \in W_x(M).$$

Hence by Mazur's theorem (or refer to the argument in the proof of [1, Lemma 2.2], [21, Proposition 3.1] or [5, Lemma 2.1]), we know

$$\boxed{\text{eq3.3}} \quad (3.3) \quad \|D^{K_1, K_2} l(\rho_x(\gamma))\|_{\mathbb{H}} \leq \sup_{r \in \mathbb{R}} |l'(r)| e^{\int_0^T \frac{|K_1(\gamma_s) + K_2(\gamma_s)|}{2} ds}, \quad \mu - a.s. \gamma \in W_x(M).$$

In addition, by the local integration by parts formula, we have

$$\boxed{\text{eq3.4}} \quad (3.4) \quad l(\rho_x) = \lim_{m \rightarrow \infty} l_x^m, \quad \mathcal{E}^{1/2} - \text{norm.}$$

13.1 Lemma 3.1. *Assume that $l \in C_0^\infty(\mathbb{R})$ with $\text{supp}(l) \subset B_{2R}(x)$, $l|_{B_R(x)} = 1$ for some $R > 0$. Then for any $p > 0$ and for any $f \in C_0^\infty(M)$,*

$$\boxed{\text{eq3.5}} \quad (3.5) \quad \lim_{T \rightarrow \infty} \frac{1}{T^p} \nabla_x \mathbb{E} \left[(1 - l(\rho_x)) f(X_T^x) \right] = 0.$$

Proof. It suffices to show that

$$\boxed{\text{eq3.6}} \quad (3.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T^p} d_x \left(\mathbb{E} \left[(1 - l(\rho_x)) f(X_T^x) \right] \right) (V) = 0, \quad \forall V \in T_x M.$$

By Lemma 2.1 in [6], we know that there exists a L^2 -integrable cut-off function $h : [0, T] \times W_x(M) \rightarrow \mathbb{R}^n$ such that

$$h(\cdot, \gamma) = 0, \quad \forall \gamma \cap B_R(x) \neq \emptyset.$$

Then by the same argument of Lemma 3.8 in [7],

$$\mathbb{E} \left[(1 - l(\rho_x)) f(X_T^x) \right]$$

is differentiable, i.e.

$$\begin{aligned} \boxed{\text{df}} \quad (3.7) \quad & d_x \left(\mathbb{E} \left[(1 - l(\rho_x)) f(X_T^x) \right] \right) (V(x)) = \mathbb{E} \left(d \left[(1 - l(\rho_x)) f(X_T^x) \right] (U_\cdot(x) h^V(x)) \right. \\ & \left. - \mathbb{E}_x \left(\left[(1 - l(\rho_x)) f(X_T^x) \right] \int_0^T \left\langle \dot{h}_r^V + \frac{1}{2} U_r^{-1}(\sigma) \text{Ric}_Z^\#(h_r^V), dB_r \right\rangle \right) \right). \end{aligned}$$

where $h^V(x) = h_\cdot(x) - h_0(x) + U_0^{-1}V$ and B_s is the stochastic anti-development of the L -diffusion processes. Thus, by (3.1), we have

$$\begin{aligned} \boxed{\text{eq3.8}} \quad (3.8) \quad & \left| d_x \left(\mathbb{E} \left[(1 - l(\rho_x)) f(X_T^x) \right] \right) (V(x)) \right| \leq \mathbb{E} \left| d \left[(1 - l(\rho_x)) f(X_T^x) \right] (U_\cdot(x) h^V(x)) \right| \\ & + \mathbb{E}_x \left| \left[(1 - l(\rho_x)) f(X_T^x) \right] \int_0^T \left\langle \dot{h}_r^V + \frac{1}{2} U_r^{-1}(\sigma) \text{Ric}_Z^\#(h_r^V), dB_r \right\rangle \right| \\ & \leq C_1 \mathbb{P}(\tau_R \leq T) \leq C_1 e^{-C_2/T} \end{aligned}$$

for some constants $C_1, C_2 > 0$. This implies (3.6). \square

Since in Theorem 1.1 we do not require that the diffusion process generated by L is non-explosive, in general, we can not establish the overall integration by parts formula. However, using the cutoff method, we may obtain the local the formula of integration by parts. The idea is to make a conformal change of metric such that the new Riemannian manifold is with bounded curvature(see [25, 5, 26]) and two metrics are the same in a compact set. In fact, for any $R > 0$, taking $\varphi \in C_0^\infty(M)$ such that $\varphi|_{B_{R+1}(x)} = 1$. Let

$$\tau_R := \inf \left\{ t \geq 0 : \rho_x(X_t^x) \geq R \right\}$$

and

$$M_\varphi := \{y \in M : \varphi(y) > 0\}, \quad L_R := g^2 L.$$

According to [20, section 2](See also [20, 13, 26, 5]) and references in, $(M_\varphi, \langle \cdot, \cdot \rangle_\varphi)$ is a complete Riemannian manifold under the metric

$$\langle \cdot, \cdot \rangle_\varphi := g^{-2} \langle \cdot, \cdot \rangle,$$

and

$$L_\varphi = \varphi^2 L = \frac{1}{2} \Delta^{(R)} + Z^{(R)}$$

for $\Delta^{(R)}$ the Laplace operator on M_φ and $Z^{(R)}$ some vector field on M_φ such that

$$\sup_{M_\varphi} (\|\text{Ric}^{(R)}\|_\varphi + \|\nabla^{(R)} Z^{(n)}\|_\varphi) < \infty,$$

where $\text{Ric}^{(R)}$, $\nabla^{(R)}$ and $\|\cdot\|_\varphi$ are the Ricci curvature, the Levi-Civita connection, and the norm of vectors on M_φ respectively. Therefore, letting \mathbb{P}_R be the distribution of the L_g -diffusion process $X^{x,R}$ on M_φ . Then, we have the Driver's formula (3.9) and the martingale representation theorem (3.11), but where the process X^x is now replaced by $X^{x,R}$. Thus, by similar computation, for any $F \in \mathcal{F}C_T$,

$$\begin{aligned} & U_0^{-1,R} \nabla_x \mathbb{E}_{\mathbb{P}_R^x} [F] \\ \text{eq3.9} \quad (3.9) \quad &= \mathbb{E} \left[D_0^{K,C} F + \int_0^T \left\{ \tilde{Q}_s^{x,R} \left(\text{Ric}_Z^R(U_s^{x,R}) - \frac{K_1(X_t^{x,R}) + C}{2} \text{Id} \right) \dot{D}_s^{K_1, K_2} F \right\} ds \right]. \end{aligned}$$

In addition, by the above construction, we know $X_t^{x,R} = X_t^x$ \mathbb{P}_x -a.s. for every $t \leq \tau_R$. Thus, by lemma 3.1 and the dominated convergence theorem, we have

$$\begin{aligned} & U_0^{-1} \nabla_x \mathbb{E}_{\mathbb{P}^x} [F] = \mathbb{E} \left[D_0^{K,C} F \right. \\ \text{eq3.10} \quad (3.10) \quad & \left. + \int_0^T \left\{ \tilde{Q}_s^x \left(\text{Ric}_Z^R(U_s^x) - \frac{K_1(X_t^x) + C}{2} \text{Id} \right) D_s^{K,C} F \right\} ds \right], \quad F \in \mathcal{F}C_{loc}^{x,R,T}. \end{aligned}$$

By the similar argument, we obtain

$$\begin{aligned} & \mathbb{E}(F(X_{[0,T]}^x) | \mathcal{F}_t) \\ \text{MF} \quad (3.11) \quad &= \mathbb{E}[F(X_{[0,T]}^x)] + \sqrt{2} \int_0^t \langle \mathbb{E}(\tilde{D}_s F(X_{[0,T]}^x) | \mathcal{F}_s), dW_s \rangle, \quad t \in [0, T], F \in \mathcal{F}C_{loc}^{x,R,T}. \end{aligned}$$

Proof of Theorem 1.1. By repeating the previous part of the proof of Theorem 1.2 and using (3.10) and (3.11), then (2)-(5) are implied by (1). Conversely, it is obvious that (2) \Rightarrow (3) and (4) \Rightarrow (5). Thus it suffices to show that (3) \Rightarrow (1) and (5) \Rightarrow (1).

(a) (3) \Rightarrow (1): Let $F \in \mathcal{F}C_{loc}^{x,R,T}$ with $F(\gamma) := l(\rho_x(\gamma))f(x) - \frac{1}{2}l(\rho_x(\gamma))f(\gamma_T)$, where $l \in C_0^\infty(\mathbb{R})$, $f \in C_0^\infty(M)$ and $\text{supp}(l) \subset B_R(x)$ and $l|_{B_{R/2}(x)} = 1$.

$$\begin{aligned} & \text{supp}(l) \subset B_R(x), \quad l|_{B_{R/2}(x)} = 1 \\ \text{eq3.12} \quad (3.12) \quad & f(x) = 0, \quad |\nabla f(x)| = 1, \quad \text{Hess}_f(x) = 0. \end{aligned}$$

By (3.1), we have

$$\text{eq3.13} \quad (3.13) \quad \left| \mathbb{E}[(l(\rho_x) - 1)\nabla f(x)] \right|^2 \leq \|\nabla f\|_\infty^2 \mathbb{P}(\tau_{R/2} \leq T) = o(T^3).$$

Combining the above inequality with Lemma 3.1, we have

$$\begin{aligned}
& \left| \mathbb{E}[l(\rho_x) \nabla f(x)] - \frac{1}{2} \nabla \mathbb{E}[l(\rho_x) f(X_T^x)] \right|^2 \\
\text{eq3.14} \quad (3.14) \quad &= \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) + \mathbb{E}[(l(\rho_x) - 1) \nabla f(x)] - \frac{1}{2} \nabla \mathbb{E}[(l(\rho_x) - 1) f(X_T^x)] \right|^2 \\
&= \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^2 + o(T^3).
\end{aligned}$$

According to (3) and (3.13),

$$\begin{aligned}
\text{eq3.15} \quad (3.15) \quad & \left| \mathbb{E}[l(\rho_x) \nabla f(x)] - \frac{1}{2} \nabla \mathbb{E}[l(\rho_x) f(X_T^x)] \right|^2 = \left| \nabla \mathbb{E}[F(X_{[0,T]}^x)] \right|^2 \\
&= \left| \nabla f(x) \mathbb{E}[l(\rho_x)] - \frac{1}{2} \nabla \mathbb{E}[l(\rho_x) f(X_T^x)] \right|^2 \\
&\leq \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| D_0^{K,C} F(X_{[0,T]}^x) \right|^2 + \int_0^T \left| \dot{D}_s^{K,C} F(X_{[0,T]}^x) \right|^2 \mu_{x,T}^{K_1, C_R^x}(ds) \right) \right] \\
&\leq \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| D_0^{K,C} [f(X_0^x) l(\rho_x)] - \frac{1}{2} D_0^{K,C} [f(X_T^x) l(\rho_x)] \right|^2 \right. \right. \\
&\quad \left. \left. + \frac{1}{4} \int_0^T \left| \dot{D}_s^{K,C} [f(X_T^x) l(\rho_x)] \right|^2 \mu_{x,T}^{K,C}(ds) \right) \right] \\
&\leq \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| l(\rho_x) \left[\nabla f(x) - \frac{1}{2} A_T^{K,C} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right] \right. \right. \right. \\
&\quad \left. \left. - \frac{1}{2} f(X_T^x) l'(\rho_x) D_s^{K,C} \rho_x \right|^2 + \right. \\
&\quad \left. \frac{1}{4} \int_0^T \left| l(\rho_x) A_T^{K_1, C_R} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) + f(X_T^x) l'(\rho_x) D_s^{K,C} \rho_x \right|^2 \mu_{x,T}^{K_1, C_R^x}(ds) \right) \right] \\
&= \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| l(\rho_x) \nabla f(x) - \frac{1}{2} A_T^{K,C} U_0^x (U_T^x)^{-1} l(\rho_x) \nabla f(X_T^x) \right|^2 \right. \right. \\
&\quad \left. \left. + \frac{\mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C}|^2 |l(\rho_x) \nabla f(X_T^x)|^2 \right) \right] + C \mathbb{P}(\tau_{R/2} \leq T) \\
&= \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| l(\rho_x) \nabla f(x) - \frac{1}{2} A_T^{K,C} U_0^x (U_T^x)^{-1} l(\rho_x) \nabla f(X_T^x) \right|^2 \right. \right. \\
&\quad \left. \left. + \frac{\mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C}|^2 |l(\rho_x) \nabla f(X_T^x)|^2 \right) \right] + o(T^3) \\
&= \mathbb{E} \left[(1 + \mu_{x,T}^{K,C}([0, T])) \times \left(\left| \nabla f(x) - \frac{1}{2} A_T^{K,C} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^2 \right. \right. \\
&\quad \left. \left. + \frac{\mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C}|^2 |\nabla f(X_T^x)|^2 \right) \right] + o(T^3).
\end{aligned}$$

Next, using the same argument of (2) \Rightarrow (1) in the proof of Theorem 1.2. we obtain (1).

(a) (5) \Rightarrow (1). Taking f and l as in (3.12), but

$$F(\gamma) := l(\rho_x(\gamma)) f(\gamma_\varepsilon) - \frac{1}{2} l(\rho_x(\gamma)) f(\gamma_T), \quad \varepsilon > 0,$$

then $F \in \mathcal{F}C_{loc}^{x,R,T}$ and by (1.8)

$$\begin{aligned} |\dot{D}_{t,s}^{K,C} F| &= \left| A_{t,\varepsilon}^{K,C} \nabla f(X_\varepsilon) - \frac{1}{2} A_{t,T}^{K,C} U_\varepsilon^x (U_T^x)^{-1} \nabla f(X_T^x) \right| 1_{[0,\varepsilon)}(s) \\ &\quad + \frac{1}{2} A_{t,T}^{K,C} |\nabla f(X_T^x)| 1_{[\varepsilon,T]}(s). \end{aligned}$$

Moreover, by (5) and (3.1), there exists constant $C_1 > 0$ depending on f, l, C, R and K such that for any $\varepsilon, T \in (0, 1)$,

$$\begin{aligned} I_\varepsilon &:= \mathbb{E} \left[\mathbb{E} \left(l(\rho_x(X_{[0,T]}^x)) f(X_\varepsilon^x) - \frac{1}{2} l(\rho_x(X_{[0,T]}^x)) f(X_T^x) \middle| \mathcal{F}_\varepsilon \right) \right]^2 \\ &\quad - \left[\mathbb{E} \left(l(\rho_x(X_{[0,T]}^x)) f(X_\varepsilon^x) - \frac{1}{2} l(\rho_x(X_{[0,T]}^x)) f(X_T^x) \right) \right]^2 \\ \text{eq3.16} \quad (3.16) \quad &\leq 2 \int_0^\varepsilon \mathbb{E} \left\{ (1 + \mu_{x,T}^{K,C}([t, T])) \left(|l(\rho_x(X_{[0,T]}^x)) \dot{D}_{t,t}^{K,C} F|^2 \right. \right. \\ &\quad \left. \left. + \int_t^T |l(\rho_x(X_{[0,T]}^x)) \dot{D}_{t,s}^{K,C} F|^2 \mu_{x,T}^{K,C}(ds) \right) \right\} dt + C\varepsilon T^4. \end{aligned}$$

Then

$$\begin{aligned} \text{eq3.17} \quad (3.17) \quad \limsup_{\varepsilon \rightarrow 0} \frac{I_\varepsilon}{\varepsilon} &\leq \mathbb{E} \left\{ l(\rho_x(X_{[0,T]}^x)) (1 + \mu_{x,T}^{K,C}([0, T])) \left(\left| \nabla f(x) - \frac{1}{2} A_T^{K,C} U_0^x (U_T^x)^{-1} \nabla f(X_T^x) \right|^2 \right. \right. \\ &\quad \left. \left. + \frac{l(\rho_x(X_{[0,T]}^x)) \mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C} \nabla f|^2(X_T^x) \right) \right\} + o(T^3) \end{aligned}$$

for small $T > 0$. According to (2.24), we have

$$\begin{aligned} \text{eq3.18} \quad (3.18) \quad \frac{I_\varepsilon}{\varepsilon} &= \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} + \frac{1}{4\varepsilon} \mathbb{E} \left[\{ \mathbb{E}(f(X_T^x) | \mathcal{F}_\varepsilon) \}^2 - (P_T f)^2(x) \right] \\ &\quad + \frac{\mathbb{E}[f(X_T^x) \{ P_\varepsilon f(x) - f(X_\varepsilon^x) \}]}{\varepsilon} + o(T^3) \\ &= 2 \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^2 + o(T^3). \end{aligned}$$

Combining this with (3.16), we get

$$\begin{aligned}
& 2 \left| \nabla f(x) - \frac{1}{2} \nabla P_T f(x) \right|^2 \\
& \leq \mathbb{E} \left\{ l \left(\rho_x \left(X_{[0,T]}^x \right) \right) \left(1 + \mu_{x,T}^{K,C}([0, T]) \right) \left(\left| \nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} A_T^{K,C} \nabla f(X_T^x) \right|^2 \right. \right. \\
& \quad \left. \left. + \frac{l \left(\rho_x \left(X_{[0,T]}^x \right) \right) \mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C} \nabla f|^2(X_T^x) \right) \right\} + o(T^3) \\
& = \mathbb{E} \left\{ \left(1 + \mu_{x,T}^{K,C}([0, T]) \right) \left(\left| \nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} A_T^{K,C} \nabla f(X_T^x) \right|^2 \right. \right. \\
& \quad \left. \left. + \frac{\mu_{x,T}^{K,C}([0, T])}{4} |A_T^{K,C} \nabla f|^2(X_T^x) \right) \right\} + o(T^3)
\end{aligned}$$

Using this estimate, we may obtain (1) by repeating the last part in the proof of (2) \Rightarrow (1) of Theorem 1.2. \square

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References

- [1] S. Aida, *Logarithmic derivatives of heat kernels and logarithmic Sobolev inequalities with unbounded diffusion coefficients on loop spaces*, J. Funct. Anal. 174(2000), 430–477.
- [2] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds*, Stoch. Proc. Appl. 119(2009), 3653–3670.
- [3] J. M. Bismut, *Large deviation and Malliavin Calculus*,
- [4] B. Capitaine, E. P. Hsu, M. Ledoux, *Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces*, Elect. Comm. Probab. 2(1997), 71–81.
- [5] X. Chen, B. Wu, *Functional inequality on path space over a non-compact Riemannian manifold*, J. Funct. Anal. 266(2014), 6753–6779.
- [6] X. Chen, X. M. Li, B. Wu, *Quasi-regular Dirichlet form on loop spaces*, Preprint.
- [7] X. Chen, X. M. Li, B. Wu, *Analysis on Free Riemannian Loop Space*, Preprint.

- [8] L.J. Cheng, A. Thalmaier, *Characterization of pinched Ricci curvature by functional inequalities*, *arXiv: 1611.02160*.
- [9] L.J. Cheng, A. Thalmaier, *Spectral gap on Riemannian path space over static and evolving manifolds*, *arXiv: 1611.02165*.
- [10] B. Driver, *A Cameron-Martin type quasi-invariant theorem for Brownian motion on a compact Riemannian manifold*, J. Funct.
- [11] A. L. Besse, *Einstein Manifolds*, Springer, Berlin, 1987.
- [12] S. Fang, *Inégalité du type de Poincaré sur l'espace des chemins riemanniens*, C.R. Acad. Sci. Paris, 318 (1994), 257-260.
- [13] S. Fang, F.Y. Wang and B. Wu, *Transportation-cost inequality on path spaces with uniform distance*, *Stochastic Process. Appl.* 118 (2008), no. 12, 2181C2197.
- [14] S. Z. Fang, B. Wu, *Remarks on spectral gaps on the Riemannian path space*, *arXiv:1508.07657*.
- [15] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics, Vol 19.
- [16] R. Haslhofer, A. Naber, *Ricci curvature and Bochner formulas for martingales*, *arXiv:1608.04371*.
- [17] E. P. Hsu, *Logarithmic Sobolev inequalities on path spaces over Riemannian manifolds*, *Comm. Math. Phys.* 189(1997), 9–16.
- [18] E. P. Hsu, *Multiplicative functional for the heat equation on manifolds with boundary*, *Mich. Math. J.* 50(2002), 351–367.
- [19] A. Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces*, *arXiv: 1306.6512v4*.
- [20] A. Thalmaier and F.-Y. Wang, *Gradient estimates for harmonic functions on regular domains in Riemannian manifolds*, J. Funct. Anal. 155:1(1998), 109–124.
- [21] M. Röckner and B. Schmuland, *Tightness of general $C_{1,p}$ capacities on Banach space*, J. Funct. Anal. 108(1992), 1–12.
- [22] F.- Y. Wang, *Weak poincaré Inequalities on path spaces*, *Int. Math. Res. Not.* 2004(2004), 90–108.
- [23] F.-Y. Wang, *Analysis on path spaces over Riemannian manifolds with boundary*, *Comm. Math. Sci.* 9(2011), 1203–1212.
- [24] F.- Y. Wang, *Analysis for diffusion processes on Riemannian manifolds*, World Scientific, 2014.
- [25] F.- Y. Wang and B. Wu, *Pointwise Characterizations of Curvature and Second Fundamental Form on Riemannian Manifolds*, *arXiv:1605.02447*.
- [26] F. -Y. Wang and B. Wu, *Quasi-Regular Dirichlet Forms on Free Riemannian Path and Loop Spaces*, *Inf. Dimen. Anal. Quantum Probab. and Rel. Topics* 2(2009) 251–267.